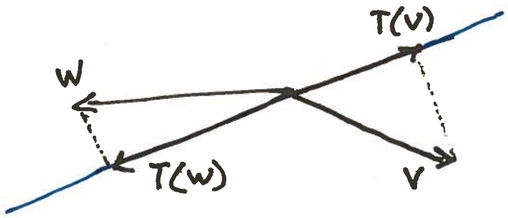
# LINEAR TRANSFORMATIONS

W/OUT COORDINATES = NO MATRIX, W/ COORDS = MATRIX

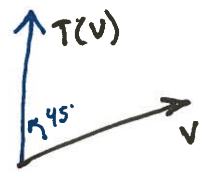
## EXAMPLE   PROJECTION

$T: R^2 \to R^2$

(T is a mapping. Takes any vector in $R^2$ a line to another line. It's a function)




ROTATION BY 45°
$T: R^2 \to R^2$




## Rules for a linear transformation

$T(V+W) = T(V) + T(W)$

$T(cV) = cT(V)$

$\to T(cV + dW) = cT(V) + dT(W)$

## EXAMPLE   MATRIX A

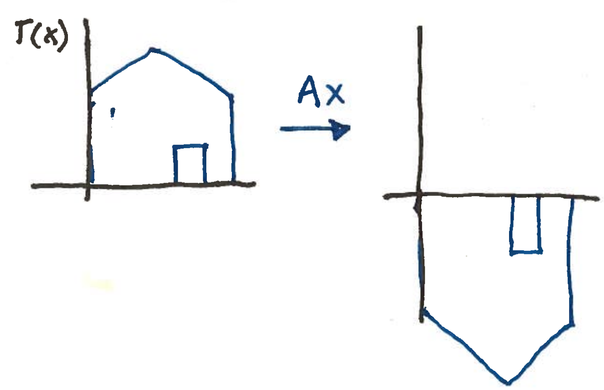
$T(V) = AV$

Let $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

Is it linear? YES ✓

$A(v+w) = Av + Aw$

$A(cV) = cAV$



Start w/ linear transformation, T

$T: R^3 \to R^2$

For examples $T(v) = Av$ where A is $2\times3$ for $T: R^3 \to R^2$
so the input v is in $R^3$, output T is in $R^2$

How much info. do we need to determine $T(v)$ for all v?

Need $T(v_1), T(v_2), \dots T(v_n)$ for any (input) <u>basis</u> $v_1, \dots, v_n$
so that we can arrive at any value in our space by a linear combination of the basis vectors

$$T(v) = c_1 T(v_1) + \dots + c_n T(v_n)$$

<u>Coordinates</u> come from a basis

- <u>Coordinates</u> of $v = c_1 v_1 + \dots + c_n v_n$

  they tell us how much of each basis vector is in v

- we have always assumed a standard othonormal basis:

$$v = \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 4\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

  but this doesn't have to be the case!

- We need to ask "what are the coordinates (coefficients) AND what is the basis (vectors)".

Construct a matrix A that represents a linear transformation, T

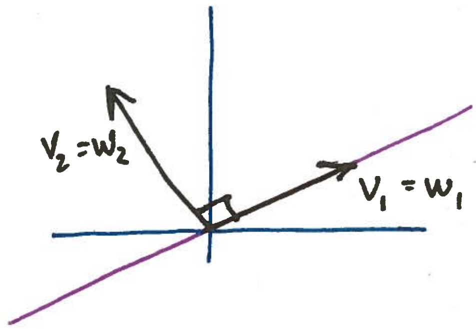
$T: R^n \to R^m$

1. Choose a basis $v_1, \dots, v_n$ for inputs $\mathbb{R}^n$
2. choose a basis $w_1, \dots, w_n$ for outputs $\mathbb{R}^m$

" We want to take vector v and express it in terms of basis vectors using coordinates $(v = c_1[b_1] + \dots + c_n[b_n])$ then multiply input coordinates by matrix A to get output coords."

Let's illustrate this idea using PROJECTION

→ All input vectors are projected onto a line $T: \mathbb{R}^2 \to \mathbb{R}^2$




- I will choose a new, non-standard basis that is the same for both inputs and outputs (hint: use eigenvalues)
- Let's solve for $A$,

$$v = c_1 v_1 + c_2 v_2$$
$$\downarrow$$
$$T(v) = c_1 v_1$$

(In projection here we only care about amount in direction $v_1$)

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$T(v) = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}}_{\text{input coords}} = \underbrace{\begin{bmatrix} c_1 \\ 0 \end{bmatrix}}_{\text{output coords}}$$

* Eigenvalue basis leads to diagonal matrix $A = \Lambda$

- Repeat projection onto 45° line using standard basis

$$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = w_1, \quad v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = w_2$$

$$P = \frac{aa^T}{a^Ta} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

this is handy but not diagonal, so clumsy.

## RULES TO FIND MATRIX A (given input/output bases)

1. $1^{st}$ column of $A$

   Apply Transformation $T(v_1) = a_{11} w_1 + a_{21} w_2 + \dots + a_{m1} w_m$

2. $2^{nd}$ column of $A$

   Apply transformation to input basis vector 2

   $T(v_2) = a_{12} w_1 + a_{22} w_2 + \dots + a_{m2} w_m$

...

## EXAMPLE LINEAR TRANSFORMATION : DERIVATIVE

$T = \frac{d}{dx}, \quad T: \mathbb{R}^3 \to \mathbb{R}^2$

Input: $c_1 + c_2 x + c_3 x^2$ Basis: $1, x, x^2$

Output: $c_2 + 2c_3 x$ Basis: $1, x$

Let's find the matrix $A$ so that

$$A \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} c_2 \\ 2c_3 \end{bmatrix}$$

We can do this by inspection

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

which works given these coordinates and bases