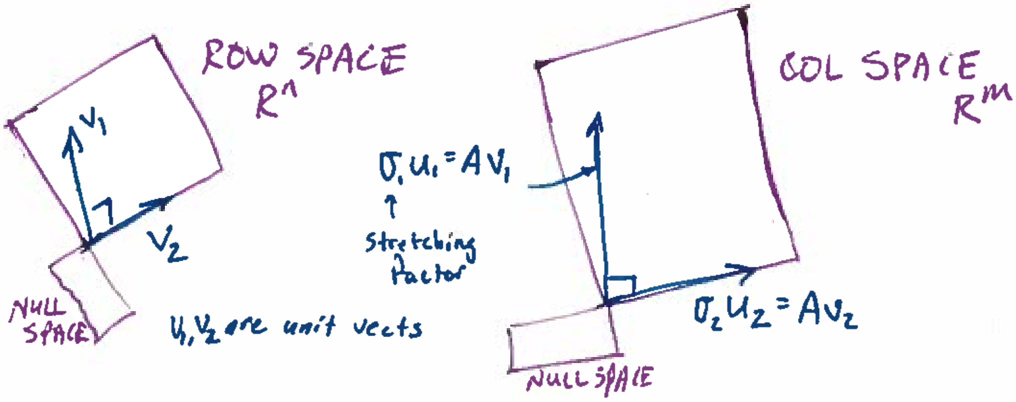
SINGULAR VALUE DECOMPOSITION [SVD]

$A = U\Sigma V^T$, Σ diagonal, U, V orthogonal

FOR AN $n \times m$ MATRIX WE LOOK AT ROW/COL SPACE [LECT 14]



- We want an orthogonal basis in row space to map to an orthogonal basis in the column space.

$$A \begin{bmatrix} v_1 & v_2 & \cdots & v_r \end{bmatrix} = \begin{bmatrix} u_1 & u_2 & \cdots & u_r \end{bmatrix} \begin{bmatrix} \sigma_1 & & \\ & \sigma_2 & \\ & & \sigma_r \end{bmatrix}$$

IN MATRIX FORM... $AV = U\Sigma \longrightarrow A = U\Sigma V^{-1} = U\Sigma V^T$ (bc square, orthogonal)

<u>EXAMPLE</u>

$$A = \begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix}$$

We will look for:

- v_1, v_2 in rowspace R^2
- u_1, u_2 in colspace R^2
- $\sigma_1 > 0$, $\sigma_2 > 0$

- u_1, u_2, v_1, v_2 orthonormal
- $Av_1 = \sigma_1 u_1$, $Av_2 = \sigma_2 u_2$

We want to get rid of U temporarily to solve for V. $A = U\Sigma V^T$

$$A^TA = V\Sigma^T U^T U \Sigma V^T$$
$$= V \begin{bmatrix} \sigma_1^2 & & \\ & \ddots & \\ & & \sigma_r^2 \end{bmatrix} V^T$$

Where v's are the eigenvectors for matrix A^TA, σ's ar eigenvalues (rank r is dimension)

If we want to get rid of V's, multiply AA^T

$$AA^T = U\Sigma V^T V \Sigma^T U^T$$
$$= U \begin{bmatrix} \sigma_1^2 & & \\ & \ddots & \\ & & \sigma_r^2 \end{bmatrix} U^T$$

So $\underline{U}$ is the <u>eigenvector matrix</u> of $\underline{AA^T}$
and $\underline{\Sigma}$ is the <u>eigenvalue diagonal matrix</u> of $\underline{AA^T}$

LET'S DO OUR EXAMPLE, $A = \begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix}$

$$A^T A = \begin{bmatrix} 4 & -3 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix} = \begin{bmatrix} 25 & 7 \\ 7 & 25 \end{bmatrix}$$

Our eigenvectors / eigen values are

$x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\lambda_1 = 32$ so $\begin{bmatrix} 25 & 7 \\ 7 & 25 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 32\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $\lambda_2 = 18$ so $\begin{bmatrix} 25 & 7 \\ 7 & 25 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = 18\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Normalizing we get

$\underline{x_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}}$, $\underline{x_2 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}}$ $\underline{\lambda_1 = 32}$, $\underline{\lambda_2 = 18}$

AND $V = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$

$$AA^T = U\Sigma\Sigma^T U^T$$

$\begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix}\begin{bmatrix} 4 & -3 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 32 & 0 \\ 0 & 18 \end{bmatrix}$ $\lambda_1 = 32$, $x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\lambda_2 = 18$, $x_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

So we found

$$U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad V = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}, \quad \Sigma = \begin{bmatrix} \sqrt{32} & 0 \\ 0 & \sqrt{18} \end{bmatrix}$$

Putting the pieces together,

$$A = U\Sigma V^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{32} & 0 \\ 0 & \sqrt{18} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

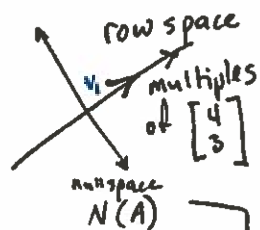
EXAMPLE UNFINISHED

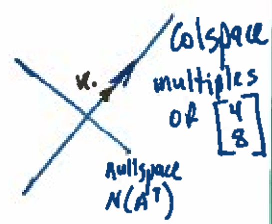
EXAMPLE 2

$$A = \begin{bmatrix} 4 & 3 \\ 8 & 6 \end{bmatrix}$$

THIS MATRIX IS SINGULAR

$$A^TA = \begin{bmatrix} 80 & 60 \\ 60 & 45 \end{bmatrix} \qquad \lambda_1 = 0, \ \lambda_2 = 125$$





$$v_1 = \begin{bmatrix} 4/5 \\ 3/5 \end{bmatrix} \qquad u_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

nullspace vector

$$\underset{A}{\begin{bmatrix} 4 & 3 \\ 8 & 6 \end{bmatrix}} = \underset{U}{\begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ 2/\sqrt{5} & -1/\sqrt{5} \end{bmatrix}} \underset{\Sigma}{\begin{bmatrix} \sqrt{125} & 0 \\ 0 & 0 \end{bmatrix}} \underset{V^T}{\begin{bmatrix} 4/5 & 3/5 \\ 3/5 & -4/5 \end{bmatrix}}$$

A = COLS × DIAG × ROWS

SUMMARY

$v_1, \ldots, v_r$ is an orthonormal basis for row space

$u_1, \ldots, u_r$ is an col space

$v_{r+1}, \ldots, v_n$ nullspace of A

$u_{r+1}, \ldots, u_m$ nullspace of A^T

WE SEE DIMENSION OF ROWSPACE / COLSPACE = RANK, r

DIMENSION OF NULLSPACE IS $n-r$ / $m-r$