A^TA IS POSITIVE DEFINITE

SIMILAR MATRICES $B = M^{-1}AM$, JORDAN FORM

Positive Definite means $x^TAx > 0$ (except for $x=0$)

For any $m \times n$ matrix A we know A^TA is square, symmetric.
Is this matrix A^TA positive definite? RANK$(A^TA) = n$

$$x^T(A^TA)x = (Ax)^T(Ax) \geqslant 0$$

YES, A^TA IS POSITIVE (SEMI) DEFINITE

Matrices A, B $(n \times n)$ are SIMILAR

means for some M, $B = M^{-1}AM$

In the eigenvalue section we said $S^{-1}AS = \Lambda$
Which we now say A is similar to Λ

SIMILAR MATRICES HAVE THE SAME EIGENVALUES !

EXAMPLE

FOR A MATRIX $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ AND ANY M, SAY $M = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}$

WE KNOW FOR A, $\lambda_1 = 1$, $\lambda_2 = 3$, $\Lambda_A = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$

$$B = M^{-1}AM = \begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & -15 \\ 1 & 6 \end{bmatrix}$$ and $\Lambda_B = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$

We start to construct a family of matrices that share the same eigenvalues. To see this

IF $Ax = \lambda x$ AND $B = M^{-1}AM$

THEN $AMM^{-1}x = \lambda x$,

$M^{-1}AMM^{-1}x = \lambda M^{-1}x$

$B M^{-1}x = \lambda M^{-1}x$

SO A, B have the same eigenvalues but different eigenvectors

Bad case:

- Repeated eigenvalues then we may not have a full set of eigenvectors and we may not be able to diagonalize.

- consider the case $\lambda_1 = \lambda_2 = 4$

↳ One small subset is the diagonal matrix $A_1 = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$ but note $M^{-1}A_1M = A_1$

↳ The big family is all other matrices w/ $\lambda = 4$ $A_2 = \begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix}$, ← only 1 eigenvector ← Jordan Form

$A_3 = \begin{bmatrix} 5 & 1 \\ -1 & 3 \end{bmatrix}$, $A_4 = \begin{bmatrix} 4 & 0 \\ 10 & 4 \end{bmatrix}$