REVIEW

① $Q = [q_1 \cdots q_n]$

$Q^T Q = I$ ← means cols are orthonormal

PROJECTIONS / LEAST SQUARES / GRAM-SCHMIDT ← basis to orthonormal basis

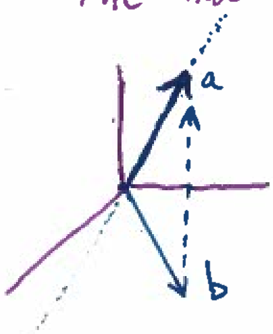
② DETERMINANTS
- properties 1-3
- Big formula ($n!$ terms)
- cofactors and A^{-1}

③ EIGENVALUES
- $Ax = \lambda x$
- $\det(A - \lambda I) = 0$
- diagonalize $S^{-1}AS = \Lambda$
- powers A^k

OLD EXAM QUESTIONS

#1

1. Given $a = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$, find proj. matrix P that projects onto the line through a, that is Pb



Formula: $P = A(A^T A)^{-1} A^T$ (LECT 15)

for a line: $P = \dfrac{a a^T}{a^T a} = \dfrac{1}{9} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} [2\ 1\ 2]$

$$\boxed{P = \frac{1}{9} \begin{bmatrix} 4 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 4 \end{bmatrix}}$$

2. What's the rank of P?
   RANK(A) = 1 → 1 independent col

3. What are the eigenvalues?
   Rank 1, Singular so $\lambda_{1,2} = 0, 0$
   By trace (P) , $\lambda_3 = 1$

4. Eigenvectors?
   a is the eigvect for $\lambda = 1$

5. Solve difference eqn $u_{k+1} = Pu_k$, $u_0 = \begin{bmatrix} 9 \\ 9 \\ 0 \end{bmatrix}$, find u_k

$$u_1 = Pu_0 = \frac{a\,a^T u_0}{a^T a}, \quad a = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \quad P = \frac{1}{9}\begin{bmatrix} 4 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 4 \end{bmatrix}$$

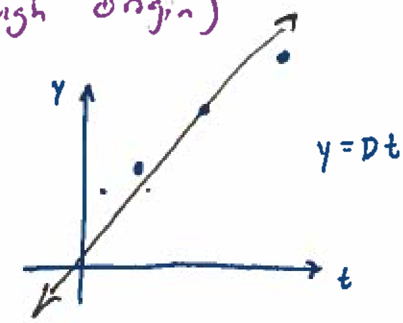
$$= a\,\frac{27}{9} = \underline{3a} = \begin{bmatrix} 6 \\ 3 \\ 6 \end{bmatrix} = \underline{u_1}$$

- we don't really need to go on bc for a projection matrix we know that $P^k = P$ so $u_2 = u_1 \ldots$
- If we didn't know this we would have to find eigenvalues, eigenvectors, and coefficients.

#2

1. Fit a straight line to points (through origin)

Pts. → (1, 4) (2, 5) (3, 8) (t, y)



$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} D = \begin{bmatrix} 4 \\ 5 \\ 8 \end{bmatrix} \leftarrow Ax = b$$

TO FIND BEST "D": (LECT 15-16)

$$A^T A \hat{D} = A^T b$$

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \hat{D} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} 4 \\ 5 \\ 8 \end{bmatrix}$$

$$14\hat{D} = 38 \rightarrow \underline{\hat{D} = \frac{38}{14}}$$

BEST FIT LINE THROUGH ORIGIN IS $\boxed{y = \frac{38}{14} t}$

P <u>projects</u> b onto the column space (line) of A

2. Given 2 vectors $a_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $a_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ find 2 orthogonal vectors in plane a_1 & a_2

We find a vector B st $B \perp a_1$ (By Gram-Schmidt)

$$B = b - \frac{A^T b}{A^T A} A \qquad \text{(LECT 17)}$$

$$= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{a_1^T a_2}{a_1^T a_1} a_1$$

where we let: $b = a_2$, $A = a_1$

$$B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{6}{14} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

← B is orthogonal to a_1

#3

1. 4×4 matrix w/ eigenvalues $\lambda_1 - \lambda_4$.
What is the condition on λ's so that matrix is invertible?

THE MATRIX IS INVERTIBLE IF ALL EIGENVALUES ARE NON-ZERO $\lambda_i \neq 0$

What is det A^{-1}

$$\text{DET}(A^{-1}) = \frac{1}{\lambda_1 \lambda_2 \lambda_3 \lambda_4}$$

What is trace of $A + I$?

$$\text{TRACE}(A+I) = (\lambda_1 + 1) + (\lambda_2 + 1) + \ldots = \sum_{i=1}^{4} \lambda_i + 4$$

#4 Given a 4×4 tridiagonal matrix

Say $D_n = \mathrm{DET}(A_n)$

$$A_4 = \begin{bmatrix} \underline{1} & \underline{1} & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Use cofactors to show [LECT 19]

$$D_n = \underline{1}\, D_{n-1} + \underline{-1}\, D_{n-2}$$

DID NOT FOLLOW LOGIC

$$A_1 = [1] \qquad A_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \qquad A_3 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \qquad A_4 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$D_1 = 1 \qquad D_2 = 0$

$$D_3 = 1 \cdot D_2 - 1 \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1 \cdot D_2 - 1 \cdot D_1 = -1$$

$$D_4 = 1 \cdot D_3 - 1 \cdot \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix} = 1 \cdot D_3 - 1 \cdot D_2 = -1$$

OK, I AGREE,

$$\boxed{D_n = D_{n-1} - D_{n-2}}$$ for tridiagonal matrices

LET'S REWRITE AS A SYSTEM

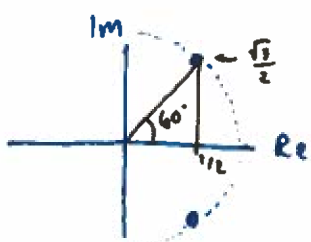
$$\begin{bmatrix} D_n \\ D_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} D_{n-1} \\ D_{n-2} \end{bmatrix}$$

FIND EIGVALS

$$\begin{vmatrix} 1-\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - \lambda + 1 = 0 \longrightarrow \lambda = \frac{1 \pm \sqrt{-3}}{2}$$

$$\left.\begin{aligned} \lambda_1 &= \frac{1+\sqrt{3}\,i}{2} \\ \lambda_2 &= \frac{1-\sqrt{3}\,i}{2} \end{aligned}\right\} *$$

ARE WE STABLE ?



$\lambda_1 = \cos(60^\circ) + i\sin(60^\circ)$

$\lambda_2 = \cos(60^\circ) - i\sin(60^\circ)$

$|\lambda_1| = |\lambda_2| = 1$

PERIODIC

#5

$$A_4 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 2 & 0 & 3 \\ 0 & 0 & 3 & 0 \end{bmatrix} \qquad A_3 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix}$$

Find projection matrix P onto col. space

$$P = A(A^TA)^{-1}A^T$$

Find eigenvalues and eigenvectors of A3.

$$|A_3 - \lambda I| = \begin{vmatrix} -\lambda & 1 & 0 \\ 1 & -\lambda & 2 \\ 0 & 2 & -\lambda \end{vmatrix} = -\lambda^3 + 5\lambda = 0$$

$$\lambda(-\lambda^2+5) \qquad \underline{\lambda_1 = 0}, \ \underline{\lambda_2 = \sqrt{5}}, \ \underline{\lambda_3 = -\sqrt{5}}$$

Find the projection matrix onto columnspace of A_4?

IF A_4 is <u>invertible</u> (not singular) then $P = I$
and forms a basis in R^4

$\text{DET}(A_4) = 9$ NOT ZERO SO INVERTIBLE
[ALSO, ROWS/COLS ARE INDEPENDENT]