MARKOV MATRICES, STEADY STATE, FOURIER SERIES

MARKOV MATRICES

① ALL ENTRIES ≥ 0

② ALL COLS ADD TO 1

In diff eqs we saw $\lambda=0$ led to steady state bc $e^{\lambda t}=1$
Here, with matrix powers we see that $\lambda=1$ is steady state bc $A^k u_0 = c_1\lambda_1^k x_1$

KEY POINTS

- $\lambda = 1$ will always be an eigenvalue for Markov matrices by ②
- All other eigenvalues of Markov matrix will be $|\lambda_i| < 1$
  - ↳ As a result we can easily pick out steady state soln
    $u_k = A^k u_0 = c_1\lambda_1^k x_1 + \ldots c_n\lambda_n^k x_n$
    - ↳ let $\lambda_1 = 1$ and all other $\lambda_i < 1$ then ~~and~~ as k increases only the λ_1 term doesn't go to zero
    - ↳ $u_k = c_1\lambda_1 x_1$ (for k big) STEADY STATE

EXAMPLE (to see why we get $\lambda=1$ when cols add to 1)

$$A = \begin{bmatrix} .1 & .01 & .3 \\ .2 & .99 & .3 \\ .7 & 0 & .4 \end{bmatrix}$$

Assuming 1 is an eigenvalue, $\lambda=1$ and we calculate $\rightarrow$ $A-\lambda I = \begin{bmatrix} -.9 & .01 & .3 \\ .2 & -.01 & .3 \\ .7 & 0 & -.6 \end{bmatrix}$

NOTE: ALL COLS OF A-I ABOVE ADD TO ZERO SO THIS MATRIX IS SINGULAR, THAT IS ALL COLS AND ROWS OF MATRIX ARE DEPENDENT.

PROOF: THAT A-I IS SINGULAR

- ROWS OF A-I ARE DEPENDENT. THIS IS CLEAR BC $X=[1\ 1\ 1]$ IS IN NULLSPACE OF A^T (THAT IS, ROWS SUM TO ZERO)
- COLS OF A-I ARE DEPENDENT BC THE EIGENVECTOR (FOR $\lambda=1$) IS IN NULLSPACE OF A (THAT IS, COLS SUM TO ZERO)

FACT: EIGENVALS OF A = EIGENVALS OF A^T

- Recall $\det(A) = \det(A^T)$

$$\det(A-\lambda I)=0 \rightarrow \det(A^T-\lambda I)=0$$

- Therefore the eigenvals are the same

APPLICATION OF MARKOV

$u_{k+1} = A u_k$, let A be a Markov Matrix

Here A describes the population of 2 States

$$\begin{bmatrix} u_{CAL} \\ u_{MASS} \end{bmatrix}_{t=k+1} = \cancel{\begin{bmatrix} u_{CAL} \\ u_{MASS} \end{bmatrix}} \begin{bmatrix} .9 & .2 \\ .1 & .8 \end{bmatrix} \begin{bmatrix} u_{CAL} \\ u_{MASS} \end{bmatrix}_{t=K}$$

It makes sense that cols add to 1 because the total pop. stays the same, but people may move between states.

- 90% of ppl stay in CAL but 10% move to MASS.
- 80% of ppl stay in MASS but 20% move to CAL.

INIT. COND. AT $t=0$ $\rightarrow$ $\begin{bmatrix} u_{CAL} \\ u_{MASS} \end{bmatrix}_0 = \begin{bmatrix} 0 \\ 1{,}000 \end{bmatrix}$

AFTER 1 STEP at $t=1$ $\rightarrow$ $\begin{bmatrix} u_{CAL} \\ u_{MASS} \end{bmatrix}_1 = \begin{bmatrix} 200 \\ 800 \end{bmatrix}$

TO SEE HOW THIS SYSTEM CHANGES OVER TIME WE NEED TO CALCULATE THE EIGVALS + EIGVECTS

- THERE WILL BE 2 λ's FOR 2×2 MATRIX
  - ↳ WE KNOW $\lambda_1 = 1$
  - ↳ THEN $\lambda_2 = .7$ (BC $\sum_i \lambda_i = TR(A)$)

- EIGENVECTS

$\lambda_1 = 1 \rightarrow \begin{bmatrix} -.1 & .2 \\ .1 & -.2 \end{bmatrix} x_1 = 0$ $\quad x_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ ← STEADY STATE

$\lambda_2 = .7 \rightarrow \begin{bmatrix} .2 & .2 \\ .1 & .1 \end{bmatrix} x_2 = 0$ $\quad x_2 = \begin{bmatrix} -1 \\ +1 \end{bmatrix}$

NOW WE CAN WRITE THE SOLN BY SOLVING

$$u_k = c_1 \lambda_1^k x_1 + c_2 \lambda_2^k x_2$$

USING INIT COND TO SOLVE FOR c_1, c_2

$$u_k = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 (.7)^k \begin{bmatrix} -1 \\ +1 \end{bmatrix}$$

$$u_0 = \begin{bmatrix} 0 \\ 1000 \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

WE SEE THAT $c_1 = \frac{1000}{3}$; $c_2 = \frac{2000}{3}$

SO OUR SOLN IS

$$\boxed{u_k = \frac{1000}{3} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \frac{2000}{3} (.7)^k \begin{bmatrix} -1 \\ 1 \end{bmatrix}}$$

TO OUR SYSTEM

$$\boxed{\begin{bmatrix} u_c \\ u_m \end{bmatrix}_{k+1} = \begin{bmatrix} .9 & .2 \\ .1 & .8 \end{bmatrix} \begin{bmatrix} u_c \\ u_m \end{bmatrix}_k , \quad \begin{bmatrix} u_c \\ u_m \end{bmatrix}_0 = \begin{bmatrix} 0 \\ 1000 \end{bmatrix}}$$

* Note, EE's often write cols as used here as rows so that $A_{LA}^T = A_{EE}$

PROJECTIONS W/ ORTHONORMAL BASIS $q_1, \cdots, q_n$

- The q's form a basis so we can express any vector in the space as

$$v = x_1 q_1 + x_2 q_2 + \cdots + x_n q_n$$

- Another way of stating this is we are expanding the vector v in the basis

- HOW DO I SOLVE FOR x_1?
  - ↳ Since I have an orthonormal basis, $q_i \cdot q_j = 0$
    so,

$$q_1^T v = x_1 \underbrace{q_1^T q_1}_{=1} + 0 + \cdots + 0$$

$$x_1 = q_1^T v$$

Matrix Notation →

$$\begin{bmatrix} \uparrow & & \uparrow \\ q_1 & \cdots & q_n \\ \downarrow & & \downarrow \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = v$$

$$Qx = v$$

$$x = Q^{-1} v$$

  - ↳ In matrix form

$$x = Q^{-1} v = Q^T v$$

FOURIER SERIES

- Fourier series relies on these facts about orthonormal bases
- We want something like

★ $$f(x) = a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \cdots$$

- where now we have functions, not matrices and we go to infinite dimensional space. The basis vectors are functions as well! And they are <u>orthogonal functions</u>.

FOR VECTORS (DOT PRODUCT)

$$V^T W = V_1 W_1 + \dots + V_n W_n$$

FOR FUNCTIONS (DOT PRODUCT)

$$f^T g = \int_0^T f(x)\, g(x)\, dx$$

- Now we have a defn for the inner product that extends the finite basis from matrices to the infinite dimensional space of continuous functions.

- How do we get $\underline{a_1}$?
  - ↳ Apply same procedure used in matrix case. multiply both sides by basis vector of choice. Here, the basis vector for $\underline{a_1}$ is $\underline{\cos x}$. Then integrate.

$$\int_0^{2\pi} f(x) \cos(x)\, dx = a_1 \int_0^{2\pi} (\cos x)^2\, dx$$

Solving,

$$a_1 = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos(x)\, dx$$

This is exactly an expansion in an orthonormal basis.