DIFFERENTIAL EQNS $\frac{du}{dt} = Au$

EXPONENTIAL e^{At} of a matrix

EXAMPLE

$$\frac{du_1}{dt} = -u_1 + 2u_2$$

$$\frac{du_2}{dt} = u_1 - 2u_2$$

INIT. COND. $u(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\xrightarrow{\text{RE-WRITE}}$ $A = \begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix}$, $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$

Let's find eigenvalues & eigenvectors...

- First we note A is singular so we know 1 eigenvalue for sure is $\lambda_1 = 0$.
- Because the sum of the eigenvalues equals the trace of A we have $\lambda_2 = -3$
- We would get the same results using our procedure. Note:

$$|A-\lambda I| = \begin{vmatrix} -1-\lambda & 2 \\ 1 & -2-\lambda \end{vmatrix} = (1+\lambda)(2+\lambda) - 2 =$$

$$= \lambda^2 + 3\lambda = \lambda(\lambda+3) = 0$$

so... $\lambda_1 = 0$, $\lambda_2 = -3$

- Now let's get our eigenvectors
  - $\lambda_1 = 0 \rightarrow \begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} \uparrow \\ x_1 \\ \downarrow \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $x_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ $(Ax_1 = 0x_1)$
  - $\lambda_2 = -3 \rightarrow \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \uparrow \\ x_2 \\ \downarrow \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $(Ax_2 = -3x_2)$

Our solution takes the form

$$u(t) = c_1 e^{\lambda_1 t} x_1 + c_2 e^{\lambda_2 t} x_2$$

Let's check this by plugging in

$$\frac{du}{dt} = Au \ldots \rightarrow \lambda_1 e^{\lambda_1 t} x_1 = A e^{\lambda_1 t} x_1$$

* this is a valid soln because $Ax = \lambda x$ ✓

Drawing from last lecture we saw

$$u_{k+1} = A u_k \quad \rightarrow \quad u_k = c_1 \lambda_1^k x_1 + \ldots + c_n \lambda_n^k x_n$$

Here we see

$$\frac{du}{dt} = Au \quad \rightarrow \quad u = c_1 e^{\lambda_1 t} x_1 + \ldots + c_n e^{\lambda_n t} x_n$$

Going back to our example let's solve for c_1, c_2 using our initial conditions...

At $t=0$ $\quad c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = u_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

and $c_1 = 1/3$, $c_2 = 1/3$

Our Soln is $\boxed{u(t) = \frac{1}{3} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \frac{1}{3} e^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}}$ $= \underset{S}{\begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}} \underset{c e^{\lambda t}}{\begin{bmatrix} 1/3 \\ 1/3 e^{-3t} \end{bmatrix}}$

steady state (as $t \rightarrow \infty$ the other term drops out)

① WHEN DO WE GET STABLE SOLUTIONS? $(u(t) \to 0)$

- For real eigenvalues, stable when $\lambda < 0$ (in general $Re(\lambda) < 0$)
  because e^{-ct} goes to zero as t grows

② WHEN DO WE GET STEADY STATE SOLNS?

- Steady state when $\lambda_1 = 0$ and others have $\lambda < 0$

③ SOLUTIONS BLOW UP WHEN...

- Soln blows up when $Re(\lambda) > 0$

Again, go back to example...

- The system $\frac{du}{dt} = Au$ is coupled but
- Solving $u = Sv$ decouples the system (by diagonalization)

$$\frac{du}{dt} = Au \quad , \quad \overset{\text{let}}{u = Sv} \qquad (S \text{ is eigenvector matrix})$$

then $$\frac{dv}{dt} = S^{-1} A S v = \Lambda v$$

so that...

$$\frac{dv_1}{dt} = \lambda_1 v \quad , \quad \frac{dv_2}{dt} = \lambda_2 v \quad , \quad \ldots$$

* Note, each equation above is decoupled!

$$v(t) = e^{\Lambda t} v(0)$$

$$u(t) = S e^{\Lambda t} S^{-1} u(0) = e^{At} u(0)$$

$$\text{so...} \quad e^{At} = S e^{\Lambda t} S^{-1}$$

WHAT IS THE MATRIX EXPONENTIAL, e^{At}?

- use power series of exponential

$$e^{At} = I + At + \frac{(At)^2}{2} + \frac{(At)^3}{6} + \dots + \frac{(At)^n}{n!}$$

THIS IS OUR NICE TAYLOR SERIES

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad \text{(POWER SERIES)}$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \qquad \text{(GEOMETRIC SERIES)}$$

LOOK AT GEO. SERIES

$$(I - At)^{-1} = I + At + (At)^2 + \dots + (At)^n$$

* this is one way to approximate a matrix inverse (as long as eigenvals of matrix At are less than 1)

The point is we can apply these familiar operations to matrices although we normally use functions

WE ARE STILL TRYING TO SHOW THAT

$$u(t) = S e^{\Lambda t} S^{-1} u(0) = e^{At} u(0)$$

using POWER SERIES ($A = S\Lambda S^{-1}$ FACT)

$$e^{At} = I + \underbrace{S\Lambda S^{-1}}_{A} t + \frac{(S\Lambda S^{-1})^2 t^2}{2} + \cdots$$

$$= \underset{\uparrow = SS^{-1}}{I} + S\Lambda S^{-1} t + \tfrac{1}{2} S\Lambda^2 S^{-1} t^2 + \tfrac{1}{6} S\Lambda^3 S^{-1} t^3$$

$$= SS^{-1}\left(I + \tfrac{1}{2}\lambda t + \tfrac{1}{6}\lambda^2 t^2 + \cdots \tfrac{1}{n!}\lambda^n t^n\right)$$

$$= S e^{\Lambda t} S^{-1} \quad \text{by applying power series}$$

So we have $\boxed{e^{At} = S e^{\Lambda t} S^{-1}}$

Given $\Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$, $e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{bmatrix}$

this is nice and decoupled now!

EXAMPLE: SETTING UP 2nd ORDER SYSTEM

Given $y'' + by' + ky = 0$

$$u = \begin{bmatrix} y' \\ y \end{bmatrix}, \quad u' = \begin{bmatrix} y'' \\ y' \end{bmatrix} = \begin{bmatrix} -b & -k \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y' \\ y \end{bmatrix}$$

For higher order systems we get $\begin{bmatrix} \vdots & \ddots & \vdots \end{bmatrix} \begin{bmatrix} \downarrow \end{bmatrix}$