LECTURE 22 | 7/8/2019

DIAGONALIZE MATRIX $S^{-1}AS = \Lambda$

POWERS OF A , EQN $u_{k+1} = A u_k$

Previous lecture on Eigenvalues:

$A - \lambda I$ singular

$Ax = \lambda x$ (x is eigenvector, λ is eigenvalue)

Today we will look at

- $S^{-1}AS = \Lambda$ · Λ is diagonal eigenvalue matrix
- where S is matrix of eigenvectors and is invertible, so we need n independent eigenvectors

Suppose we have n lin. indep. eigenvectors of A, put them in columns of S.

$$AS = A\begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ x_1 & x_2 & \cdots & x_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix} = \begin{bmatrix} \uparrow & & \uparrow \\ \lambda_1 x_1 & \cdots & \lambda_n x_n \\ \downarrow & & \downarrow \end{bmatrix}$$

$$= \underset{\uparrow \text{ Original matrix, } S}{\begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ x_1 & x_2 & \cdots & x_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}} \underset{\uparrow \text{ Eigen value Matrix, } \Lambda}{\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & 0 \\ \vdots & \vdots & & \vdots \\ & & & \lambda_n \end{bmatrix}} = S\Lambda$$

So far we see $\boxed{AS = S\Lambda}$

With S invertible, $\boxed{\begin{array}{l} S^{-1}AS = \Lambda \\ A = S\Lambda S^{-1} \end{array}}$

Example POWERS OF A

If $Ax = \lambda x$ then $A^2 x = \lambda A x = \lambda^2 x$

Using our new formula,

$$A^2 = S \Lambda S^{-1} S \Lambda S^{-1} = S \Lambda^2 S^{-1}$$

and,

$$A^k = S \Lambda^k S^{-1}$$

WE WILLSEE THE STRENGTH OF DIAGONALIZATION IS

$$\Lambda^k = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}^k = \begin{bmatrix} \lambda_1^k & & \\ & \ddots & \\ & & \lambda_n^k \end{bmatrix}$$

* ONE STEP TO DO POWERS

A will have n independent eigenvectors (and is diagonalizable) if all the λ's are different (that is, no repeated eigenvalues).

Example : TRIANGULAR MATRIX

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \quad \ldots \quad \det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 1 \\ 0 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 , \quad \underline{\lambda = 2, 2} \text{ Eigenvalue}$$

$$A - 2I = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x = 0, \quad \underline{x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}} \text{ Eigenvector}$$

(only 1, wanted 2)

- Triangular matrices are tough to work with, cannot diagonalize!
- Here we did not have distinct eigenvalues so only 1 eigenvector

Equation : $u_{k+1} = A u_k$

Start with given vector u_0 and multiply by A each iteration.

$$u_1 = A u_0 , \quad u_2 = A^2 u_0 , \quad \ldots \quad u_k = A^k u_0$$

- To solve u_0 we write it as a linear combination of eigenvectors

(because n unique eigenvectors form a basis for R^n ... so we can express u_0 as lin. comb. of eig-vectors)

$$u_0 = c_1 x_1 + c_2 x_2 + \ldots c_n x_n \qquad = S c$$

then multiplying by A (and recall $Ax = \lambda x$)

$$A u_0 = c_1 \lambda_1 x_1 + c_2 \lambda_2 x_2 + \ldots + c_n \lambda_n x_n$$

doing this k times

$$A^k u_0 = c_1 \lambda_1^k x_1 + c_2 \lambda_2^k x_2 + \ldots + c_n \lambda_n^k x_n = \boxed{\Lambda^k S c = u^k}$$

* correction $u_k = S \Lambda^k c$

Now we have this nice formula, let's do an example.

FIBONACCI :

$F_0 = 0, \quad F_1 = 1 \qquad [0, 1, 1, 2, 3, 5, 8, 13, \ldots]$

$F_{k+2} = F_{k+1} + F_k$ "Standard" form of Fibonacci Eqn

Let's put this in a more familiar form

make into a system

$F_{k+2} = F_{k+1} + F_k$

$F_{k+1} = F_{k+1}$

Let $u_k = \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix}$

then $u_{k+1} = \begin{bmatrix} F_{k+2} \\ F_{k+1} \end{bmatrix}$

$$u_{k+1} = \underset{\uparrow \atop A}{\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}} \underset{\uparrow \atop u_k}{\begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix}}$$

So, $\boxed{u_{k+1} = A\,u_k}$

where, $\boxed{A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \; u_k = \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix}}$

FIBONACCI SEQUENCE IN LIN. ALGEBRA FORM

$u_k = A^k u_0$

NOTE, WE CAN STOP HERE.

$u_0 = \begin{bmatrix} F_1 \\ F_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ so $u_k = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^k \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

AND $F_k = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^k \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

but the k power is not efficient

Let's Find the EIGENVALUES

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - \lambda - 1 = 0 \qquad \lambda = \frac{1 \pm \sqrt{5}}{2}$$

$$\boxed{\lambda_1 = \frac{1+\sqrt{5}}{2}, \quad \lambda_2 = \frac{1-\sqrt{5}}{2}}$$

From the eqn $u^k = \lambda^k S c$ we see that the EIGENVALUE controls the growth of this function.
Here, $\lambda_1 > 1$ while $\lambda_2 < 1$ so λ_1 term will dominate as k increases.

Let's find the EIGENVECTORS

$$A - \lambda_n I = \begin{bmatrix} 1-\lambda_n & 1 \\ 1 & -\lambda_n \end{bmatrix} x_n = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\boxed{x_1 = \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix}, \quad x_2 = \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix}}$$

Let's put this all together

$$u_0 = \begin{bmatrix} F_1 \\ F_0 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{u_0}$$

we need to solve

$$C_1 X_1 + C_2 X_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$C_1 \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix} + C_2 \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\underline{C_1 = \frac{1}{\lambda_1 - \lambda_2}}, \quad \underline{C_2 = \frac{1}{\lambda_2 - \lambda_1}} \quad (C_2 = -C_1) \qquad \lambda_1 - \lambda_2 = \sqrt{5}$$

S is our EIGENVECTOR MATRIX, Λ is our EIGENVALUE IDENTITY MATRIX

$$S = \begin{bmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 1 & 1 \\ \uparrow & \uparrow \\ x_1 & x_2 \end{bmatrix}, \quad \Lambda = \begin{bmatrix} \overset{\lambda_1}{\frac{1+\sqrt{5}}{2}} & 0 \\ 0 & \underset{\lambda_2}{\frac{1-\sqrt{5}}{2}} \end{bmatrix}$$

C is our vector of coefficients

$$C = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ -1/\sqrt{5} \end{bmatrix}$$

Our final formula /system is :

$$\cancel{u_k = \frac{1}{\sqrt{5}} \begin{bmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{bmatrix}^k}$$

$$F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{n} - \left(\frac{1-\sqrt{5}}{2}\right)^{n}\right.$$

$$u_k = \begin{bmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{bmatrix}^k \begin{bmatrix} 1/\sqrt{5} \\ -1/\sqrt{5} \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_1^{k+1}/\lambda_1-\lambda_2 - \lambda_2^{k+1}/\lambda_1-\lambda_2 \\ \lambda_1^{k}/\lambda_1-\lambda_2 - \lambda_2^{k}/\lambda_1-\lambda_2 \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} \lambda_1^{k+1} - \lambda_2^{k+1} \\ \lambda_1^{k} - \lambda_2^{k} \end{bmatrix}$$