Orthogonal Basis $q_1, \cdots, q_n$, Gram-Schmidt $A \to Q$

Orthogonal Matrix Q ,

$$Q = \begin{bmatrix} | & & | \\ q_1 & \cdots & q_n \\ | & & | \end{bmatrix} \qquad Q^TQ = \begin{bmatrix} - & q_1^T & - \\ & \vdots & \\ - & q_n^T & - \end{bmatrix} \begin{bmatrix} | & & | \\ q_1 & \cdots & q_n \\ | & & | \end{bmatrix} = I$$

if Q is square then if $Q^TQ = I$ the $Q^T = Q^{-1}$. Only square matrices can be orthogonal / orthonormal.

<u>Example</u>

$$Q = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{so} \quad Q^TQ = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

so Q is an <u>orthogonal matrix</u>.

<u>Example</u>

Given matrix Q with orthonormal cols. <u>Project</u> onto its column space

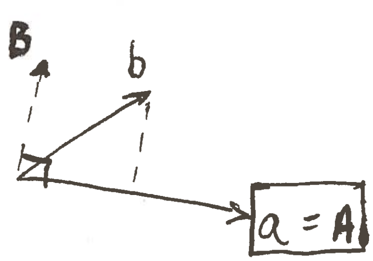
$$P = Q(Q^TQ)^{-1}Q^T = QQ^T \qquad (\text{because } Q^TQ = I)$$

$$[\, = I \,] \qquad (\text{if } Q \text{ is square})$$

<u>Projection matrix</u> is symmetric and $(QQ^T)(QQ^T) = QQ^T$

Gram-Schmidt

Given vectors a, b (independent), I want orthogonal vectors A, B



$$\boxed{B = b - \frac{A^Tb}{A^TA} A}$$

and orthonormal vectors

$$q_1 = \frac{A}{|A|}, \quad q_2 = \frac{B}{|B|}$$

If we had a 3rd independent vector c, then

$$\boxed{C = c - \frac{A^Tc}{A^TA} A - \frac{B^Tc}{B^TB} B}$$

Gram-Schmidt Example

$$A = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$a = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

$$B = \underbrace{\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}}_{b} - \frac{3}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

So now $A = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$ and $A \perp B$