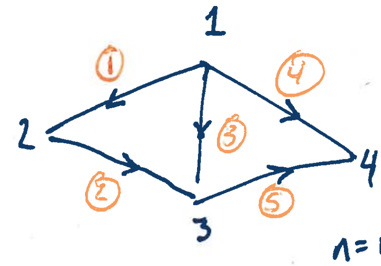
# Graphs, Networks

- Graphs have nodes and edges
- We can represent the graph by a matrix. The number of rows equals the number of edges (m). Num cols = Num nodes




$n = 4$ nodes
$m = 5$ edges

## Incidence Matrix:

$$A = \begin{array}{c} \text{Nodes} \rightarrow \\ \text{Edges} \downarrow \end{array} \begin{array}{cc} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ & \left[\begin{array}{cccc} -1 & +1 & 0 & 0 \\ 0 & -1 & +1 & 0 \\ -1 & 0 & +1 & 0 \\ -1 & 0 & 0 & +1 \\ 0 & 0 & -1 & +1 \end{array}\right] \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}$$

loop 1 (edges 1, 2, 3)

loop 2 (edges 3, 4, 5)

Loops correspond to linearly dependent rows! $[R_1 + R_2 = R_3]$

Nullspace: tells us how to combine cols to get zero

$\underline{Ax = 0}$

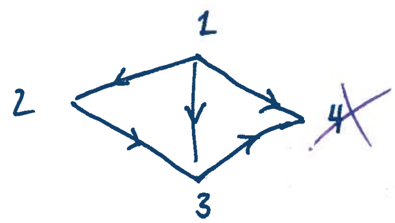
$$\begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0 \Rightarrow \begin{bmatrix} x_2 - x_1 \\ x_3 - x_2 \\ x_3 - x_1 \\ x_4 - x_1 \\ x_4 - x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Nullspace vector $= c \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

$\dim[N(A)] = 1$

Back to our graph:

if we set the node to zero (ground), we can remove the column associated w/ the node.




## Properties of our graph (matrix):

rank$(A) = 3$ rank$(A^T) =$

$$A^T = \begin{bmatrix} -1 & 0 & -1 & -1 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

$\dim(N(A)) = 1$

$$A^T y = \begin{bmatrix} -1 & 0 & -1 & -1 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix}$$

$\underline{\dim(N(A^T))} = m - r$
$= 5 - 3 = \underline{2}$

## $A^T y = 0$:

$$-y_1 - y_3 - y_4 = 0$$

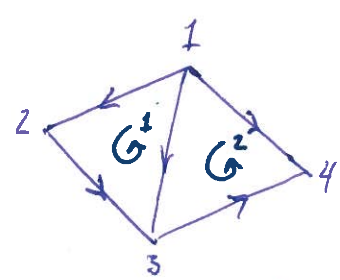
(Sum of currents into node 1 must be equal to zero!)

$$y_1 - y_2 = 0$$
$$y_2 + y_3 - y_5 = 0$$
$$y_4 + y_5 = 0$$

## Basis for $N(A^T)$:

2 vectors in nullspace of $A^T$ correspond to 2 closed loops!

$$\underset{1}{\begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}} \quad \underset{2}{\begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 1 \end{bmatrix}}$$




the current around big loop $(1, 1, 0, -1, 1)$ is a linear comb.

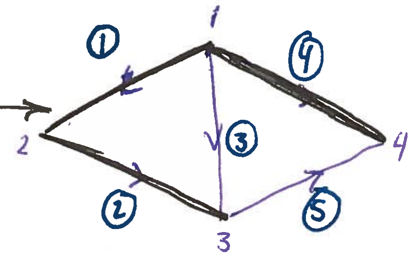
## Row Space of A / column space of $A^T$

$$A^T = \begin{bmatrix} -1 & 0 & -1 & -1 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

pivot cols          free cols

"loops" in graph lead to dependencies. Note col 3 is dependent on cols 1, 2. This is clear from the nullspace vector: $(1, 1, -1, 0, 0)$

The pivot cols form a graph w/out a loop shown in black to the right. This is called a tree!




## Nullspace revisited

$$\dim(N(A^T)) = m - r = \#\text{ of independent loops}$$
$$= \#\text{edges} - (\#\text{nodes} - 1)$$

recall, $\text{rank} = n - 1$

$$\boxed{\#\text{nodes} - \#\text{edges} + \#\text{loops} = 1}$$ Euler's Formula

# CIRCUITS STEPS

$x = x_1, x_2, x_3, \ldots$
(potential at nodes)

$\downarrow \; Ax = e$

$x_2 - x_1, \; x_3 - x_2, \; \ldots$
(potential differences)

$\downarrow$ ohm's Law

$y_1, y_2, y_3, \ldots$
(currents on edges)

$\downarrow \; Ce = y$

Krchoff's Current Law
$Ay = 0 \;,\; Ay = f$

Putting these steps together we can summarize these transformations:

$$\boxed{A^T C A x = f}$$

Interesting note:

$A^T A$ is always symmetric