Four fundamental subspaces

4 Subspaces:

1. column space $C(A)$
2. Null space $N(A)$
3. Row space $C(A^T)$ • All combinations of the rows of A
   = all combs of cols of A^T
4. Null Space of A^T $N(A^T)$ • "left" null space of A

Given A is $m \times n$, $C(A) \subset R^m$, $N(A) \subset R^n$, $C(A^T) = R^n$, $N(A^T) \subset R$

BASIS and DIMENSION

$C(A)$: basis is pivot cols (after row reduction)
dimension is rank$(A) = r$

$C(A^T)$: dimension is rank $(A) = r$
basis is first r rows of R (reduce row-echelon matrix)

$N(A)$: dimension is # free variables $= n-r$
basis is special soln when setting free variables

$N(A^T)$: dimension is $m-r$

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R$$

- ~~Column~~ Row operations preserve row space not col space
- $C(R) \neq C(A)$
- $C(R) = C(A^T)$

A basis for the row space ~~of~~ A or R is the 1st r rows of R

Left Nullspace

do G-J elimination

$$[A \; I] = \left[\begin{array}{cccc|ccc} 1 & 2 & 3 & 1 & 1 & 0 & 0 \\ 1 & 1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 2 & 3 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\text{rref}} [R \; E] \quad \text{so} \quad EA = R$$

do my operations on I retroactively

$$\begin{bmatrix} \cancel{1}^{-1} & \cancel{0}^{2} & 0 \\ \cancel{0}^{-1} & \cancel{1}^{-1} & 0 \\ \cancel{0}^{-1} & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} -1 & 2 & 0 \\ -1 & -1 & 0 \\ -1 & 0 & 1 \end{bmatrix} = E$$

Write out $EA = R$

$$\begin{bmatrix} -1 & 2 & 0 \\ -1 & -1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix}$$

we get nullspace by
$-1 \times row_1(A) + 0 \times row_2(A) + 1 \times row_3(A)$

The basis for the left nullspace is the last m-r rows of elimination matrix E