Complete Solution of $Ax = b$

Example:

same matrix from lecture 7

$$\begin{aligned} x_1 + 2x_2 + 2x_3 + 2x_4 &= b_1 \\ 2x_1 + 4x_2 + 6x_3 + 8x_4 &= b_2 \\ 3x_1 + 6x_2 + 8x_3 + 10x_4 &= b_3 \end{aligned}$$

let $b_1 = 1, b_2 = 5, b_3 = 6$

$$b = \begin{bmatrix} 1 \\ 5 \\ 6 \end{bmatrix}$$

re-write matrix (augmented matrix : $[A \ b]$)

pivot

$$\left[\begin{array}{cccc|c} 1 & 2 & 2 & 2 & b_1 \\ 2 & 4 & 6 & 8 & b_2 \\ 3 & 6 & 8 & 10 & b_3 \end{array}\right] \xrightarrow{\text{elimination}} \left[\begin{array}{cccc|c} 1 & 2 & 2 & 2 & b_1 \\ 0 & 0 & 2 & 4 & b_2 - 2b_1 \\ 0 & 0 & 2 & 4 & b_3 - 3b_1 \end{array}\right]$$

pivot, pivot

pivot columns

$$\left[\begin{array}{cccc|cc} 1 & 2 & 2 & 2 & b_1 & 1 \\ 0 & 0 & 2 & 4 & b_2 - 2b_1 & 3 \\ 0 & 0 & 0 & 0 & b_3 - b_2 - b_1 & 0 \end{array}\right] \rightarrow 0 = b_3 - b_2 - b_1$$

b_3 is a linear combination of b_2, b_1 so this is solvable!

Solvability (condition on b)

- $Ax = b$ is solvable when b is in column-space of A, that is b must be a combination of the columns

equivalent

- If a combination of rows of A gives a zero row, then the same comb. of the entries of b must give 0

To find solution to $Ax = b$

1. Find a particular solution

method 1: set all free variables to zero (free x_2, x_4)
solve $Ax=b$ for pivot variables (pivot x_1, x_3)

from example
$$\begin{cases} x_1 + 2x_3 = 1 \\ 2x_3 = 3 \end{cases} \rightarrow \begin{matrix} x_1 = -2 \\ x_3 = 3/2 \end{matrix} \rightarrow x_p = \begin{bmatrix} -2 \\ 0 \\ 3/2 \\ 0 \end{bmatrix}$$

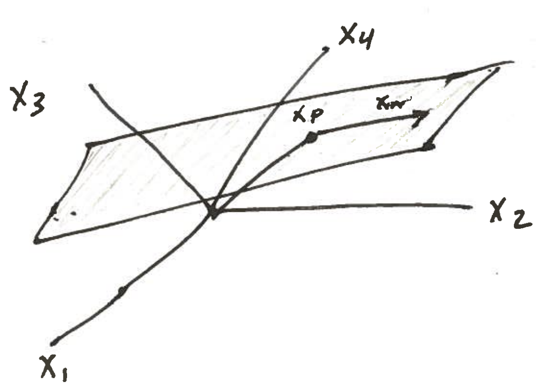
2. Find null-space solution (homogeneous) * from lecture 7

3. Write complete solution (by superposition)

$$x = \begin{bmatrix} -2 \\ 0 \\ 3/2 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} \\ \\ \\ \end{bmatrix} + c_2 \begin{bmatrix} \\ \\ \\ \end{bmatrix}$$

* recall, null-space consists of all combinations of special solutions (2 spec. solns bc 2 free vars)

* no constant multiplier for part. soln bc solves $Ax=b$ but yes for NS soln bc it solves $Ax=0$



set of solns to $Ax=b$ do <u>not</u> form a subspace.
Recall the homogeneous solution do form a subspace bc the null-space here is a 2-D subspace in R^4.

Not a subspace bc particular soln shifts away from 0.

Big Picture:

Consider $M \times N$ matrix of rank r

* rank currently defined as # pivots ($r < M$, $r \leq N$)

- FULL COLUMN RANK ($r = N$)
  - N pivots, a pivot in all columns
  - 0 free variables

$\Rightarrow$ Null(A) = zero vector

$\Rightarrow$ Soln to $Ax=b$ is now $x = x_p$ (unique soln)
(that is there are either 0 or 1 solns)

Example:

A will have 2 pivots (full col'n rank)

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \\ 6 & 1 \\ 5 & 1 \end{bmatrix} \qquad R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

if $b = [4\ 3\ 7\ 6]'$ then $x_p = [1\ 1]'$

- FULL ROW RANK ($r = M$)
  - M pivots, every row has pivot
  - can solve $Ax = b$ for <u>every</u> b (existence)
  - left with $N - r$ free variables

<u>Example</u>:

A has rank 2 (full row rank)

$$A = \begin{bmatrix} 1 & 2 & 6 & 5 \\ 3 & 1 & 1 & 1 \end{bmatrix} \qquad R = \begin{bmatrix} 1 & 0 & & F & \\ 0 & 1 & & & \end{bmatrix}$$

- FULL (ROW AND COL) RANK ($\underline{r = M = N}$)
  - these matrices are <u>invertible</u>!
  - null space is zero vector
  - can solve $Ax = b$ for every b

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$$

$R = I$ (always)

<u>Summary</u>

$r = M = N$	$r = N < M$	$r = M < N$
$R = I$	$R = \begin{bmatrix} I \\ 0 \end{bmatrix}$	$R = \begin{bmatrix} I & F \end{bmatrix}$
1 solution to $Ax = b$	0 or 1 solution	∞ solutions

* I does NOT have to come 1st

RANK tells everything about # solutions except exact entries!

$r < M$, $r < N$
$R = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$
0 or ∞ solutions